Appels Theorem for a certain 3rd order differential equation

Summary: u(x) and v(x) are independent solutions of

$$u''(x) + p(x)\, u'(x) + q(x)\, u(x) = 0, \qquad (1)$$

Then the general solution of

$$w'''(x) + 3\, p(x)\, w'' + [2\, p(x)^2 + p'(x) + 4\, q(x)]\, w' + [4p(x)\, q(x) + 2\, q'(x)]\, w = 0 \qquad (2)$$

is: $A\, p(x)^2 + B\, p(x)\, q(x) + C\, p(x)^2$, with arbitrary constans A, B, C.

E.T. Whitaker, G.N. Watson: A Course of Modern Analysis. Cambridge, University Press, 1927. p.298. Ex.10 quotes: Appel, Comptes Rendus, XCL

■ The general case, eqs.(1) and (2)

■ Appels operator

```
o3[ww_] := D[ww, {x, 3}] + 3 p[x] D[ww, {x, 2}] +
   (2 p[x]^2 + D[p[x], x] + 4 q[x] ) D[ww, x] + (4 p[x] q[x] + 2 D[q[x] , x]) ww
```

```
duv = o3[ u[x] v[x]]
```

$$u[x]\, v[x]\, (4\, p[x]\, q[x] + 2\, q'[x]) + (2\, p[x]^2 + 4\, q[x] + p'[x])\, (v[x]\, u'[x] + u[x]\, v'[x]) + 3\, v'[x]\, u''[x] + 3\, u'[x]\, v''[x] + 3\, p[x]\, (2\, u'[x]\, v'[x] + v[x]\, u''[x] + u[x]\, v''[x]) + v[x]\, u^{(3)}[x] + u[x]\, v^{(3)}[x]$$

```
duu = o3[ u[x] u[x]]
```

$$u[x]^2\, (4\, p[x]\, q[x] + 2\, q'[x]) + 2\, u[x]\, (2\, p[x]^2 + 4\, q[x] + p'[x])\, u'[x] + 6\, u'[x]\, u''[x] + 3\, p[x]\, (2\, u'[x]^2 + 2\, u[x]\, u''[x]) + 2\, u[x]\, u^{(3)}[x]$$

```
dvv = o3[ v[x] v[x]]
```

$$v[x]^2\, (4\, p[x]\, q[x] + 2\, q'[x]) + 2\, v[x]\, (2\, p[x]^2 + 4\, q[x] + p'[x])\, v'[x] + 6\, v'[x]\, v''[x] + 3\, p[x]\, (2\, v'[x]^2 + 2\, v[x]\, v''[x]) + 2\, v[x]\, v^{(3)}[x]$$

■ u[x] is solution of (1)

```
equ = o[u[x]] == 0
```

```
q[x] u[x] + p[x] u′[x] + u″[x] == 0
```

```
souu = Solve[equ, Derivative[2][u][x]] // Flatten
```

```
{u″[x] → -q[x] u[x] - p[x] u′[x]}
```

```
souuu = D[souu, x]
```

```
{u⁽³⁾[x] → -u[x] q′[x] - q[x] u′[x] - p′[x] u′[x] - p[x] u″[x]}
```

```
souuu /. souu
```

```
{u⁽³⁾[x] → -u[x] q′[x] - q[x] u′[x] - p′[x] u′[x] - p[x] (-q[x] u[x] - p[x] u′[x])}
```

■ v[x] is solution of (1)

```
eqv = o[v[x]] == 0
```

```
q[x] v[x] + p[x] v′[x] + v″[x] == 0
```

```
sovv = Solve[eqv, Derivative[2][v][x]] // Flatten
```

```
{v″[x] → -q[x] v[x] - p[x] v′[x]}
```

```
sovvv = D[sovv, x]
```

```
{v⁽³⁾[x] → -v[x] q′[x] - q[x] v′[x] - p′[x] v′[x] - p[x] v″[x]}
```

```
sovvv /. sovv
```

```
{v⁽³⁾[x] → -v[x] q′[x] - q[x] v′[x] - p′[x] v′[x] - p[x] (-q[x] v[x] - p[x] v′[x])}
```

■ u[x] u[x] is solution of (2)

```
fuu = duu /. Join[souuu, sovvv] /. Join[souu, sovv]
```

```
u[x]^2 (4 p[x] q[x] + 2 q′[x]) +
 2 u[x] (2 p[x]^2 + 4 q[x] + p′[x]) u′[x] + 6 u′[x] (-q[x] u[x] - p[x] u′[x]) +
 2 u[x] (-u[x] q′[x] - q[x] u′[x] - p′[x] u′[x] - p[x] (-q[x] u[x] - p[x] u′[x])) +
 3 p[x] (2 u′[x]^2 + 2 u[x] (-q[x] u[x] - p[x] u′[x]))
```

```
Simplify[fuu]
```

```
0
```

- **u[x] v[x]   is solution of (2)**

```
fuv = duv /. Join[souuu, sovvv] /. Join[souu, sovv]
```

```
u[x] v[x] (4 p[x] q[x] + 2 q′[x]) +
 v[x] (-u[x] q′[x] - q[x] u′[x] - p′[x] u′[x] - p[x] (-q[x] u[x] - p[x] u′[x])) +
 3 (-q[x] u[x] - p[x] u′[x]) v′[x] + 3 u′[x] (-q[x] v[x] - p[x] v′[x]) +
 (2 p[x]^2 + 4 q[x] + p′[x]) (v[x] u′[x] + u[x] v′[x]) +
 u[x] (-v[x] q′[x] - q[x] v′[x] - p′[x] v′[x] - p[x] (-q[x] v[x] - p[x] v′[x])) +
 3 p[x] (v[x] (-q[x] u[x] - p[x] u′[x]) + 2 u′[x] v′[x] + u[x] (-q[x] v[x] - p[x] v′[x]))
```

```
Simplify[fuv]
```

```
0
```

- **v[x] v[x]   is solution of (2)**

```
fvv = dvv /. Join[souuu, sovvv] /. Join[souu, sovv]
```

```
v[x]^2 (4 p[x] q[x] + 2 q′[x]) +
 2 v[x] (2 p[x]^2 + 4 q[x] + p′[x]) v′[x] + 6 v′[x] (-q[x] v[x] - p[x] v′[x]) +
 2 v[x] (-v[x] q′[x] - q[x] v′[x] - p′[x] v′[x] - p[x] (-q[x] v[x] - p[x] v′[x])) +
 3 p[x] (2 v′[x]^2 + 2 v[x] (-q[x] v[x] - p[x] v′[x]))
```

```
Simplify[fvv]
```

```
0
```

- **The special case, Mathieus eq. , p(x) ≡ 0, q(x) = J(x).**

```
suma = {p[x] → 0, p'[x] → 0, q[x] → J[x]};
```

```
o3[w[x]] /. suma
```

```
2 w[x] q′[x] + 4 J[x] w′[x] + w^(3)[x]
```