The Quasi-static Approximation for Weakly Conducting Media and Applications

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***Abstract*: The electric quasi-static approximation for weakly conducting media is explained and applied. It permits one to get dynamic results from electrostatic data. The signals induced in the electrodes of particle counters are computed by Ramo's theorem relating the induced current in the electrode to the static field (= weighting field) generated by a static field applied to the same electrode. This theorem is generalized to weakly conducting media. - The weighting fields are computed from integral representations, whose convergence has been accelerated by extracting the source singularity and those due to the first few images. This is applied to configurations comprising several layers.**

***Keywords*: Quasi-static approximation, weakly conducting media, currents induced by moving charges, convergence acceleration.**

I. Introduction

Quasi-static approximations are very useful on account of simplicity of solution methods and results; these are well known for the case of isolating dielectrics and ideally conducting boundaries. These methods have been extended to the case where materials have a small conductivity [1]. In that case one gets a partial differential equation for the potential resembling a diffusion equation. This approach permits one to obtain impedances for problems with small conductivity from purely static results [2], [3], [4]. The signals induced in electrodes by moving charges are computed by Ramo's theorem [5], [6], which is an application of the reciprocity theorem. This can be extended to weakly conducting media by applying a Laplace transform to the quasi-static equations [7].

Field evaluations may be hampered by slow convergence of series or integrals. The evaluation is improved by extracting the singularity from the integral with the help of known integrals [8], [9].

The relaxation time due to a weak conductivity of a homogeneous dielectric is:

$$\tau = \varepsilon/\sigma. \tag{1}$$

Nowadays in particle counters materials as glass or melamine-phenolic laminate are used; their relative dielectric constants are about 2 to 4; their specific resistances amount to $10^{11} - 10^{11}$ Ωcm, corresponding to conductivities of $10^{-9} - 10^{-10}$/m. The resulting relaxation times lie between 0.003s and 0.3s; these are long as compared to the duration of the signals, or the times they need to pass to the electrodes, or the time the charge cloud needs to transverse the gap [11], [13], [3], [14]. on the other hand one wants to know what is the influence of this weak conductivity on the signals. The electric quasi-static approximation is a a very useful tool for such investigations.

B.S. has become acquainted with this method at the 8-th of these conferences, [10]. Later on he was refered to the books [1],[2] devoted to these powerful approximation schemes. Both authors and their students used the method to erive results important to the theory of signal generation and modification in particle counters. Here we give a short survey of some of these applications.

II. The Quasi-static Approximation

The electric quasi-static approximation is best derived from the Laplace transform

$$\mathcal{L}[\vec{A}(\vec{x},t)] = \vec{\overline{A}}(\vec{x},s),\ \ \mathcal{L}[\frac{\partial \vec{A}(\vec{x},t)}{\partial t}] = s\vec{\overline{A}}(\vec{x},s), \tag{2}$$

(where it has been assumed that all fields and sources are zero for $t \leq 0$) of Maxwell's equations

$$\nabla\,\vec{D} = \overline{\rho},\ \ \vec{\overline{D}} = \varepsilon\vec{\overline{E}}; \qquad \nabla\vec{\overline{B}} = 0,\ \ \vec{\overline{B}} = \mu\vec{\overline{H}}; \tag{3}$$

$$\nabla\times\vec{\overline{E}} = -s\vec{\overline{B}} \qquad \nabla\times\vec{\overline{H}} = \vec{\overline{j}}_e + \sigma\vec{\overline{E}} + s\vec{\overline{D}} \tag{4}$$

for a linear isotropic medium with permittivity $\varepsilon(\vec{x},s)$ and conductivity $\sigma(\vec{x},s)$. $\vec{j}_e$ is an 'externally impressed' current that is connected with an 'external' charge density by $\nabla\vec{\overline{j}}_e = -s\overline{\rho}_e$. It is presupposed that the conductivity σ is small so that the time-dependent magnetic field is negligible and the electric field is irrotational:

$$\nabla\times\vec{\overline{E}} = -s\vec{\overline{B}} = 0 \quad \Rightarrow \vec{\overline{E}} = -\nabla\overline{\Phi}; \tag{5}$$

and by taking the divergence of the second equation in (4) we find

$$\nabla[\sigma(\vec{x},s)\nabla]\overline{\Phi}(\vec{x},s) + \nabla[\varepsilon(\vec{x},s)\nabla]s\overline{\Phi}(\vec{x},s) = -s\rho_e(\vec{x},s), \tag{6}$$

which we can write as

$$\nabla[\epsilon(\vec{x},s)\nabla]\overline{\Phi}(\vec{x},s) = -\overline{\rho}_e(\vec{x},s) \tag{7}$$

with

$$\epsilon(\vec{x},s) = \varepsilon(\vec{x},s) + \frac{1}{s}\sigma(\vec{x},s). \tag{8}$$

This equation has the same form as the Poisson equation for electrostatic problems. In order to find the potential to a given charge density pulse

$$\rho_e(\vec{x},t) = \rho(\vec{x})\delta(t) \qquad \rightarrow \qquad \overline{\rho}_e(\vec{x},s) = \rho(\vec{x})\ . \tag{9}$$

the following equation must be solved:

$$\nabla[\epsilon(\vec{x},s)\nabla]\overline{\Phi}(\vec{x},s) = -\rho(\vec{x}). \tag{10}$$

From this we can conclude the following statements:
If we know the electrostatic potential for the charge density $\rho(\vec{x})$ in a medium with given $\varepsilon(\vec{x})$ we obtain the time dependent potential for a charge density $\rho(\vec{x})\delta(t)$ in a

medium with conductivity $\sigma(\vec{x}, s)$ and permittivity $\varepsilon(\vec{x}, s)$ by replacing ε with $\varepsilon + \sigma/s$ and performing the inverse Laplace transform.

Since the Green's function for the quasi-electrostatic problem is the potential for the source $\delta(\vec{x})\delta(t)$ the same conclusion applies:

If we know the Green's function for a medium with given $\varepsilon(\vec{x})$ we obtain the time dependent Green's function for a medium with conductivity $\sigma(\vec{x}, s)$ and permittivity $\varepsilon(\vec{x}, s)$ by replacing ε with $\varepsilon + \sigma/s$ and performing the inverse Laplace transform.

In the special case where the dielectric constant and the conductivity do not depend on space and time, one may perform an approach similar to that just given starting from the time-dependent Maxwell equations. Then one obtains the following differential equation for the potential excited by a charge density ρ, which may depend on space and time:

$$\sigma\nabla^2\Phi + \varepsilon\frac{\partial}{\partial t}\nabla^2\Phi \;=\; \frac{\partial\rho}{\partial t}. \tag{11}$$

The corresponding homogeneous equation may be solved by separting time t from the space variables: $\Phi(\vec{r}, t) = \Phi(\vec{r})\; T(t)$. This gives the following differential equation for the time function $T(t)$, whose solution contains the relaxation time τ defined in Eq.(1):

$$\varepsilon\frac{dT}{dt} + \sigma\, T = 0; \tag{12}$$

$$T = T_0\; e^{-t/\tau}, \quad \Phi(\vec{r}, t) = \Phi(\vec{r})\; e^{-t/\tau}. \tag{13}$$

While the solutions of Maxwell's equation give retarded solutions, the potentials and fields obtained from Eq.(11) correspond to a propagation with infinite velocity.

III. One-Layer Problems

The above result is applied to a few simple examples.

A. Point Charge in an Infinite Medium

The Green's function for a homogeneous medium characterized by a constant dielectric constant ε is given by

$$\overline{G}(\vec{r}) = \frac{1}{4\pi\varepsilon|\vec{r}|} \tag{14}$$

Replacing ε by $\varepsilon+\sigma/s$ and performing the inverse Laplace transform we find the Green's function for a medium with constant conductivity σ and permittivity ε as

$$G(\vec{r}, t) = \frac{1}{4\pi\varepsilon|\vec{r}|}\left(\delta(t) - \frac{\sigma}{\varepsilon}e^{-\frac{t}{\tau}}\right), \qquad \tau = \frac{\varepsilon}{\sigma}. \tag{15}$$

E.g. putting at time $t = 0$ a charge density $\rho(\vec{r})$ into the medium i.e. $\rho_e(\vec{r}, t) = \rho(\vec{r})\Theta(t)$ the time dependent potential is given by

$$\begin{aligned}\Phi(\vec{r}, t) &= \int_V\int_0^t G(\vec{r}-\vec{r}', t-t')\rho(\vec{r}')\Theta(t')dt'd^3r' \\ &= \frac{e^{-\frac{t}{\tau}}}{4\pi\varepsilon}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r' \end{aligned} \tag{16}$$

The potential is equal to the electrostatic one, but 'destroyed' with the time constant $\tau = \sigma/\varepsilon$.

B. Impedance of an Infinite Plane Condensor

If an isolating dielectric is replaced by a weakly conducting one then the capacitance of the circuit is shunted by a resistance. The admittance consists of a capacitance $C = j\omega C$ and a conductance G, which is proportional to the capacitance.

$$G = C\sigma/\varepsilon, \qquad Y \;=\; G \;+\; j\omega C. \tag{17}$$

This is shown for a plane condensor. One starts from the homogeneous Eq.(11) and a time-dependence $e^{j\omega t}$ is assumed. So we get:

$$(\sigma + j\omega\varepsilon)\Delta\Phi = j\omega\epsilon\Delta\Phi = 0.$$

The lower electrode is grounded, the upper one at potential V_0, so the potential is:

$$\Phi = \frac{V_0}{D}\; z\; e^{j\omega t}.$$

The charge on an electrode of area F is:

$$Q = -F\; D_n = -\epsilon F E_n = \epsilon F\frac{V_0}{D}\; e^{j\omega t}.$$

D_n, E_n respectively are the normal components of the dielectric displacement, the electric field respectively. We get for the current:

$$I = dQ/dt = j\omega\epsilon F\frac{V_0}{D}\; e^{j\omega t} = YV_0\; e^{j\omega t}.$$

Comparing the sencond and the third member of this equation leads to the results given in Eq.(17).

IV. Two-Layer Problems

The medium is inhomogeneous inasmuch as it consists of two homogeneous half spaces with an interface at (say) $z = 0$. The relative dielectric constants and the conductivities are ε_1, σ_1 for $z < 0$; ε_2, σ_2 for $z > 0$. In Eqs.(7) to (10) these constants must get subscripts numbering the half-spaces.

Across the interface the potentials and their normal derivatives must fulfil the following continuity conditions:

$$z = 0: \quad \Phi_1 = \Phi_2, \quad \epsilon_1\frac{\partial\Phi_1}{\partial n} = \epsilon_2\frac{\partial\Phi_2}{\partial n}. \tag{18}$$

The Greens function for the potential equation consists of four pieces $G_{ik}(\vec{r}, \vec{r}')$; the first (second) subscript gives the half-space containing the point of observation (the source point). It is found with the help of image charges [12]:

$$\overline{G}_{11} \;=\; \frac{1}{4\pi\epsilon_1}\left(\frac{1}{R_1} - \frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\frac{1}{R_2}\right), \tag{19}$$

$$\overline{G}_{22} \;=\; \frac{1}{4\pi\epsilon_2}\left(\frac{1}{R_2} - \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{R_1}\right), \tag{20}$$

$$\overline{G}_{ij} \;=\; \frac{1}{4\pi}\frac{1}{\epsilon_1+\epsilon_2}\frac{1}{R_j}, \;\; (i \neq j). \tag{21}$$

Distances R_1 and R_2 are shown in Fig.1.

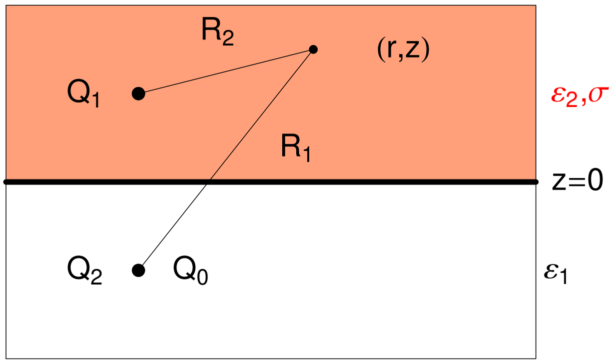

Figure 1. Distances: $R_1(R_2)$ from source (image) point to point of observation. Q_0 primary charge, $Q_1, (Q_2)$ are the images belonging to medium 1 (2).

A. *Point Charge at Interface*

At time $t = 0$ a point charge Q it put to the point $\vec{r} = 0$. So $Q\overline{G}_{ij}$ with $i \neq j$ gives the electrostatic solution ($\sigma = 0$):

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{\epsilon_1 + \epsilon_2}\frac{1}{|\vec{r}|} \tag{22}$$

This has the same form as the above solution (14). So the solution for weakly lossy media is:

$$\Phi(\vec{r}) = \frac{1}{4\pi}\frac{1}{\epsilon_1 + \epsilon_2}\frac{1}{|\vec{r}|}\, e^{-t/\tau} \tag{23}$$

with

$$\tau = \frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon 2}.$$

B. *Point Charge Changing in Time*

A point charge fixed at $r = 0, z = -z_0 < 0$ grows from 0 to Q during a time τ_0 and stays constant afterwards, see Fig.2. The charge density (for $t \geq 0$) and its transform are:

$$\begin{aligned} \rho(\vec{r}, t) &= \frac{Q}{2\pi r}\delta(r)\delta(z - z_0)Q_0(t), \\ Q_0(t) &= \frac{t}{\tau_0}\Theta(\tau_0 - t) + \Theta(t - \tau_0), \qquad (24) \\ \overline{\rho}(\vec{r}, s) &= \frac{Q}{2\pi r}\delta(r)\delta(z - z_0)\frac{1 - e^{-s\tau_0}}{s^2\tau_0}. \end{aligned}$$

The electrostatic solution is calculated with the help of the Green's function found above. Thereafter ε_i is replaced with $\epsilon_i = \varepsilon_i + \sigma_i/s$, and the inverse Laplace transform is done. The solution so obtained may be represented as generated by the original charge $Q_0(t)$, Eq.(24), and two time-dependent image charges, $Q_1(t)$ and $Q_2(t)$. Here a special case is presented, where the orginal charge is in vacuum at a fixed position in front of a weakly conducting half space; so $\epsilon_1 = \varepsilon_1,\ z < 0$ and $\epsilon_2 = \varepsilon_2 + \sigma/s,\ z > 0$. The potentials in the two half spaces are:

$$\begin{aligned} \Phi_1(r, z, t) &= \frac{1}{4\pi\varepsilon_1}\left[\frac{Q_0(t)}{R_1} + \frac{Q_1(t)}{R_2}\right], \qquad (25) \\ \Phi_2(r, z, t) &= \frac{1}{4\pi\varepsilon_1}\frac{Q_2(t)}{R_1}; \qquad (26) \\ Q_1(t) &= -\,Q_0(t) - \frac{2Q\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\frac{\tau}{\tau_0}q_h(t), \qquad (27) \end{aligned}$$

$$\begin{aligned} Q_2(t) &= \frac{2Q\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\, q_h(t), \qquad (28) \\ q_h(t) &= (1 - e^{-t\tau}) - (1 - e^{-(t-\tau_0)\tau})\Theta(t - \tau_0); \\ \tau &= (\varepsilon_1 + \varepsilon_2)/\sigma. \qquad (29) \end{aligned}$$

These expressions show that the electro quasi-static so-

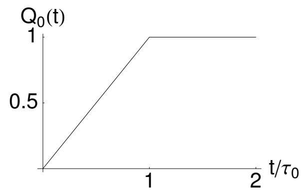

Figure 2: Time dependence of primary charge.

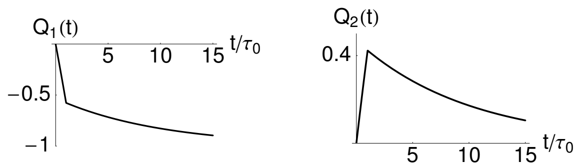

Figure 3. Time dependence of image charges for $\tau_0 \ll \tau$. $\varepsilon_1 = \varepsilon_0,\ \varepsilon_2 = 3.5\,\varepsilon_0$. $\tau_0 = 0.1\,\tau$.

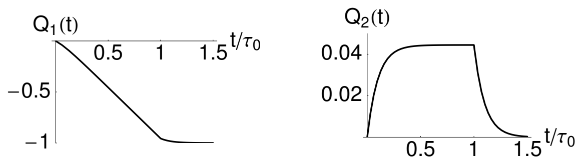

Figure 4. Time dependence of image charges for $\tau_0 \gg \tau$. $\varepsilon_1 = \varepsilon_0$, , $\varepsilon_2 = 3.5\,\varepsilon_0$. $\tau_0 = 10\,\tau$.

lution for this time-dependent problem is quite similar to that of the electrostatic problem. In both cases two image charges together with the primary charge produce the field, but in the latter case all three charges are time-depedent. At first the image charges increase as does the primary charge; thereafter the conductivity of medium 2 destroys the field in this half space so that in the end the field distribution is the same as that prevailing if the conductivity of medium 2 were infinite. The potential depends on two rates, the given growth rate τ_0 and the decay rate τ. Figs.3 and 4 show the time dependence of the image charges for large and small decay rates (as compared to the growth rate). For large decay rate the image charges follow the primary one faithfully; except that Q_1 contributing the the field in medium 1 has a sign opposite to that of Q_0. For small decay rate the charge Q_1 shows the same behavior, while $Q_2(T)$ approximates the time derivative of Q_0.

C. *A Moving Point Charge*

A point charge Q starts at time $t = 0$ at $r = 0, z = -z_0 < 0$ and runs with constant velocity $\vec{e}_z v$ towards the interface $z = 0$ separating the two media.The charge density and its transform are:

$$\begin{aligned} \rho(\vec{r}, t) &= \frac{Q}{2\pi r}\,\delta(r)\delta(z - z_0 - vt), \qquad (30) \\ \overline{\rho}(\vec{r}, s) &= \frac{Q}{2\pi r v}\,\delta(r)\Theta(z - z_0)\, e^{-s(z-z_0)/v}. \qquad (31) \end{aligned}$$

In particular, it is assumed that medium 1, in which the charge moves, is vacuum, while medium 2 is a weakly conducting dielectric. The solution method described in the preceding subsection leads to the following potentials:

$$\Phi_1(r,z,t) = \frac{1}{4\pi\varepsilon_1}\left[\frac{Q}{R_1(t)} + \frac{Q_1}{R_2(t)} + \frac{\varepsilon_1}{\varepsilon_2}\Phi_{1nl}\right], \quad z \le 0; \quad (32)$$

$$\Phi_2(r,z,t) = \frac{1}{4\pi\varepsilon_2}\left[\frac{Q_2}{R_1(t)} + \Phi_{2nl}\right], \quad z \ge 0; \quad (33)$$

$$Q_1 = Q\,\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad (34)$$

$$Q_2 = \frac{2Q\varepsilon_2}{\varepsilon_1 + \varepsilon_2}; \quad (35)$$

$$R_{1,2}(t) = \sqrt{r^2 + (z \mp z_0 \mp vt)^2}, \quad (36)$$

$$R_3(t) = R_1(t); \quad (37)$$

$$\Phi_{inl}(r,z,t) = -\frac{Q_2}{\tau}\int_0^t dt'\,\frac{e^{-(t-t')/\tau}}{R_{i+1}(t')}. \quad (38)$$

The above representation shows that the field belonging to a moving constant charge may be more complex than that due to a changing charge with fixed position. In fact, also the moving charge induces moving images but with constant charge; however, there is an additional non-local term embracing the history of the field, i.e. the losses due to the weak conductivy σ of medium 2.

Some graphs should convey an overview over the influence of the electrical parameters. In Fig.5 the magnitude of the relaxation time is shown for various values of the relative dielectric constants and of the conductivity.

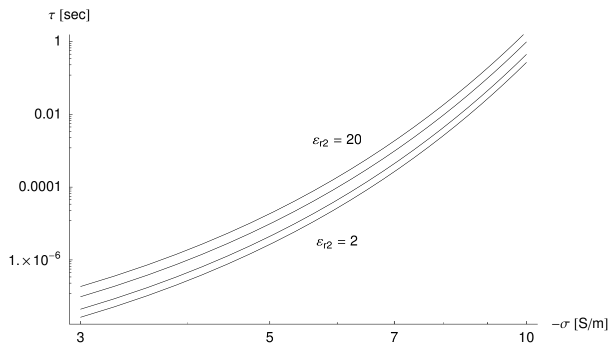

Figure 5. The relaxation time τ, Eq.(), versus conductivity σ of medium 2 for various values of the relative dielectric constant $\varepsilon_{r2}(= 2, 4, 10, 20)$; $\varepsilon_{r1} = 1$..

As long as the non-local term is small as compared to those stemming from the images the latter give a good approximation and a very simple and compact description of the field. In order to get a feeling for the magnitude of this non-local term, we shall endeaver to make a numeric estimate for this term in Eq.(33). We extract the first term of the square bracket; in the resulting square bracket the first term equals unity. The second term of the square bracket is then:

$$\mathcal{I} = -\beta\, e^{-\beta\gamma}\sqrt{\rho^2 + (\zeta + 1 - \gamma)^2}\ \times$$
$$\times \int_0^\gamma d\alpha\, e^{\alpha\beta}/\sqrt{\rho^2 + (\zeta + 1 - \alpha)^2}.$$

Here new dimensionless variables have been introduced:

$$\alpha = t'/\beta\tau,\ \beta = |z_0|/v\tau,\ \rho = r/|z_0|,\ \zeta = z/|z_0|.$$
$$t = \gamma|z_0|/v, \quad 0 \le \gamma < 1.$$

So we get for eq.(33):

$$\Phi_2(r,z,t) = \frac{1}{2\pi(\varepsilon_1 + \varepsilon_2)}\frac{1}{R_1(t)}\,[1 - |\mathcal{I}|]. \quad (39)$$

For $r \sim \rho = 0$ the integral may be evaluated analytically:

$$|\mathcal{I}(\beta,\gamma,\zeta)]| = (\zeta + 1 - \gamma)\beta e^{-\beta(\zeta+1-\gamma)} \times \quad (40)$$
$$\times\ [\Gamma(0, \beta(1 - \gamma + \zeta) - \Gamma(0, \beta(1 + \zeta))]\,;$$
$$\Gamma(a,z) = \int_z^\infty t^{a-1}\, e^{-t}\, dt. \quad (41)$$

$\Gamma(a,z)$ is the incomplete Gamma function (function program available in *Mathematica*).

As long as $\mathcal{I}$ is small as compared to unity, the first term depending only on the image charge and the distance gives a good approximation to the total potential. So we give some graphs for $|\mathcal{I}(\beta,\gamma,\zeta)|$. It is seen that the non-local

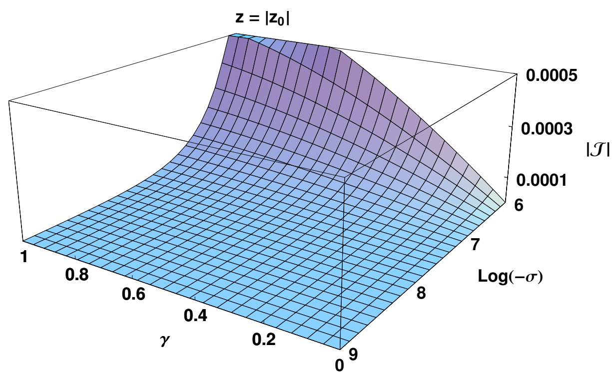

Figure 6. $|\mathcal{I}(\beta,\gamma,\zeta)]|$versus γ (= dimensionless time) and the conductivity σ.

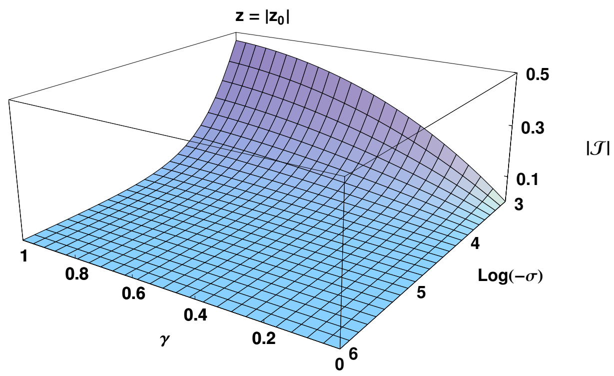

Figure 7. $|\mathcal{I}(\beta,\gamma,\zeta)]|$versus γ (= dimensionless time) and the conductivity σ.

term is small if the conductivity σ is smaller than 10^{-4}S/m. But if the conductivity becomes larger than this value, the non-local term grows quite fast.

V. Signals in Counters

The signal currents I induced in grounded electrodes by moving charges (as e.g. in particle counters) are calculated with the help of Ramo's theorem, [5], [6]:

$$I = q_0 \vec{v} \cdot \vec{E}/V$$

(q_0 charge moving with low velocity $\vec{v}$, $\vec{E}$ weighting field = electric field generated by voltage V applied to the electrode in the absence of the inducing charge). This theorem entails a considerable simplification of the task to calculate the current flowing from or to a counter electrode as it is induced by a charge moving in the device. To see this, imagine a point charge slowly moving in a simple plane condensor; one want's to calculate the current flowing from one plate. Without Ramo's theorem, one must solve the boundary value problem to get the field generated by the charge (distribution) in the device; the field component normal to the electrode integrated over the whole electrode gives the charge or the current, if the charge is moving. Using Ramo's theorem one needs just the homogeneous field excited in the empty condensor by the voltage applied to the electrodes.

A. Extension of Ramo's theorem to weakly conducting media

Ramo's theorem [5], [6] is valid for non-lossy media only. It has been generalized to the case of dielectrics with a small conductivity and a number of applications have been considered, [7]. This generalization is:
The Voltage induced by a dime-depedent charge distribution $\rho(\vec{x},t)$ on an electrode embedded in a medium of permittivity $\varepsilon(\vec{x},s)$ and conductivity $\sigma(\vec{x},s)$ can be calculated in the following way: remove the charge, apply a delta current $q_0\delta$ on the electrode in question which defines a time-depedent potential $\psi(\vec{x},t)$ (called the weighting potential) in the space between the electrodes from which $V(t)$ can be calculated according to:

$$V(t) = \frac{1}{q_0}\int_0^t \int_V \psi(\vec{x}',t-t')\,\frac{\partial\rho(\vec{x}',t')}{\partial t'}\,d^3x'\,dt'. \quad (42)$$

For small values of the relative D.K. (ϵ_r), the changes introduced by the real part of the dielectric constant are of the order of $(\epsilon_r-1)/(\epsilon_r+1)$, [13].

The changes due to the conductivity are of the order of $D\epsilon/v\sigma$, (σ,ϵ conductivity, D.K. of the medium, v velocity of the inducing electron (or ion) avalanche,D layer width, [11].

In the treatment of resistive plate chambers containing several layers the method described permitted W.R.[7] to show rigorously the influence of the conductivity on the signal. A highly conducting layer attached to an electrode becomes just part of that electrode. A floating (i.e. not grounded) highly coducting layer does not screen off the field. Opposite charges accumulate at opposing sides of that layer; and so the field is transmitted. These are obvious things, but they are rarely proved by explicit calculations.

VI. Covergence acceleration of integral representations of Green's functions

For some kinds of field computations one may evaluate analytic field representations containing Green's functions numerically. In most boundary value problems it is not possible to get closed expressions representing the Green's functions. So one must use integral representations or infinite series of particular solutions obtained by solving the potential equation by separation of variables. The Green's function is singular at $\vec{r}=\vec{r}'$ but regular elsewhere, in principle. However, in the series or integral representations one encounters a memory effect due to the singularity: There are curves or surfaces passing through the singular point, on which the series or integrals fail to converge although there is $\vec{r}\neq\vec{r}'$. Even in the neighbourhood of these manifolds convergence will be slow, too slow for numerical evaluations. In many cases there is a remedy. One subtracts from the integral functions having the same asymptotic behavior as the deleterious terms but which are so simple that the corresponding integrals over the sanitizing functions can be evaluate analytically. This remedy is displayed below in two examples, where we deal only with the Green's functions.

A. Plane condensor containing one medium

The Green's function of the potential equation $\Delta G = -\delta(\vec{r}-\vec{r}')$ fulfilling the boundary conditions $z=0,D\ \vee\ \rho\to\infty:\ G=0$, may be represented in cylindrical coordinates ρ,ϕ,z as:

$$\begin{aligned} &G(\rho,\phi,z;\rho',\phi',z') = \quad (43)\\ &\quad = \frac{1}{\pi D}\sum_{n=0}^{\infty} \sin(k_n z)\ \sin(k_n z')\ K_0(k_n P);\\ &P = (x-x')^2+(y-y')^2\\ &\quad = \rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'),\\ &k_n = n\pi/D. \quad (44)\end{aligned}$$

It is obvious that each term of the series (43) is singular on the line $P=0\ \Leftrightarrow\ \rho=\rho'$. The integral representation

$$\begin{aligned} &G(\rho,\phi,z;\rho',\phi',z') = \quad (45)\\ &= \frac{1}{2\pi}\int_0^\infty d\kappa\ J_0(\kappa P)\ g(\kappa;z_<,z_>),\\ &g(\kappa;z_<,z_>) = \frac{\sinh(\kappa z_<)\ \sinh[\kappa(D-z_>)]}{\sinh(\kappa D)} \quad (46)\end{aligned}$$

runs into trouble in the following three planes: 1. $z=z'$ 2. $z=z'=0$ 3. $z=z'=D$. The physics of these three cases is: 1. source and observation point are in the same plane; The obervation point and the image point below the conducting plane $z=0$ coalesce; 3. Similarly, but at the plane $z=D$. This is seen by rewriting the amplitude (46) as:

$$\begin{aligned} g(\kappa;z_<,z_>) &= Z/N; \quad (47)\\ Z &= e^{-\kappa(z_>-z_<)} - e^{-\kappa(2D-z_>-z_<)}\\ &\quad -e^{-\kappa(z_>+z_<)} - e^{-\kappa(2D-z_>+z_<)},\\ N &= 1-e^{-2\kappa D}.\end{aligned}$$

It is obvious that the exponent of one of the four exponentials in the numerator Z becomes zero in the three cases just listed. The square root κ occuring in the denominator of the aymptotic representation of the Bessel function together with the oscillations still permits convergence, but a prohibitvely slow one. This is illustrated in Fig.8.

The remedy is to subtract from $g(\kappa;z_<,z_>)$ the three dangerous exponentials:

$$\begin{aligned} g_s &= 2g - [e^{-\kappa(z_>-z_<)} - e^{-\kappa(z_>+z_<)}\\ &\quad -e^{-\kappa(2D-z_>-z_<)}]\end{aligned}$$

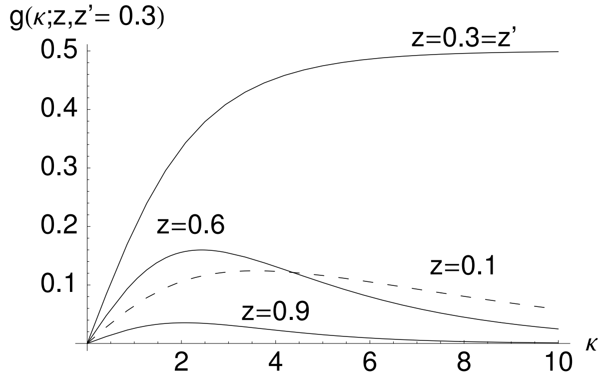

Figure 8. Dependence of the amplitude function g, Eq.(46),47, of Eq.(45) on the integration variable κ for fixed z' and various z. For $z = z'$ g no longer decreases as $\kappa \to \infty$.

$$
\begin{aligned}
&= Z_s/N; \qquad (48)\\
Z_s &= e^{-\kappa(2D-z+z')} + e^{-\kappa(2D+z-z')}\\
&\quad -e^{-\kappa(2D+z+z')} - e^{-\kappa(4D-z-z')}.
\end{aligned}
$$

This new amplitude function g_s converges fast for any values of z and z' in the interval $[0, D]$. The integrals containing the dangerous exponentials times the Bessel function $J_0(\kappa P)$ can be done by Sommerfelds integral. So the completely sanitized representation of the Greens function is:

$$
\begin{aligned}
&G(\rho,\phi,z;\rho',\phi',z') = \qquad (49)\\
&= \frac{1}{4\pi\sqrt{P^2+(z-z')^2}} - \frac{1}{4\pi\sqrt{P^2+(z+z')^2}}\\
&\quad - \frac{1}{4\pi\sqrt{P^2+(2D-z-z')^2}}\\
&\quad + \frac{1}{4\pi}\int_0^\infty d\kappa\, J_0(\kappa P)\, g_s(\kappa;z,z').
\end{aligned}
$$

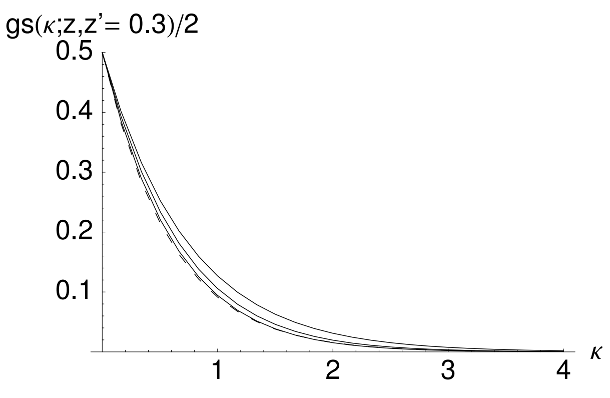

Figure 9. Dependence of the amplitude function g_s, Eq.(48), of Eq.(49) on the integration variable κ for fixed $z' = 0.3$ and the same z's as in Fig.8.

More details for the derivations and evaluations described here very briefly have been presented in a report [8]. There the computation times needed for the evaluations of the two representations (43) and (49) are compared.

The method is also useful in applications where difference methods are used. It is difficult to represent a point charge in a mesh. The Green's function yields a representation of the field due to this point charge. Further the Green's function and the corresponding field components must be evaluated in the neighbourhood of the singularity. The methods jsut described helps in these evaluations.

B. Plane condensor containing two plane media

The integral representation of the Greens function for a plane condensor filled with two dielectrics has been treated by the method explained in the preceding subsection. Besides the planes: 1. $z = z'$ 2. $z = z' = 0$ 3. $z = z' = D$ listed above also the plane of the interface between the two media has a deleterious effect on the convergence; so additional image charges must be extracted from the integrand. The integral representations so sanitized have been used in simulations studying the influence of space charge on the growth of the charge avalanche in a resistive plate chamber, [15].

VII. Conclusions

Quasi-static methods are very useful in applied electromagnetics for obtaining formulas permitting semi-analytic evaluations; the expressions are somewhat involved, but their numeric evaluation is fast and not difficult. The books by Haus and Melcher [1] and by Fano, Chu and Adler [2] are excellent references for these approximation schemes.

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