Elektrodynamics Exercises WS 2015-16

Problem set 2, 06.11.2015

Problem 1: Wave packet in vacuum

The propagation of a wave packet in vacuum is given by

$$E(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \mathrm{Re}\left[E(k) e^{i(kx-\omega_k t)}\right]$$

where $\omega_k = c|k|$.
Consider now a wave packet, where the amplitude at $t = 0$ is given by the distribution

$$E(k) = \frac{2E_0}{\Delta k} e^{-\frac{(k-k_0)^2}{2\Delta k^2}},$$

Determine the Form of the wave packet as function of x and t. **Please calculate integral explicitly!**
Hint: Rewrite integral in the Gaussian form and take into account that

$$\int_{-\infty}^{\infty} dk e^{-b(x-a)^2} = \sqrt{\frac{\pi}{b}}$$

for any a and $\mathrm{Re}[b] > 0$.

Problem 2: Reflection and Refraction at Boundaries

Consider the following situation with two dielectrics. One with dielectric constant ϵ_1 for $y < 0$, and the other with dielectric constant ϵ_2 for $y > 0$. The permeabilities are $\mu_1 = \mu_2 = \mu_0$. The boundary between the two dielectrics is the plane $y = 0$. A monochromatic electromagnetic wave hits the boundary from below ($y < 0$). The wave has **E**-field of the form

$$\mathbf{E} = \mathbf{E}_I e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}.$$

The momentum $\mathbf{k}$ lies in the x-y-plane and has an angle α with the y axis. $\mathbf{E}_I$ is also in the x-y-plane ($\mathbf{E}_I$ parallel to the plane of incidence).

a) Use the relation between **E** and **D** for a monochromatic plane wave, the refraction law, as well as the boundary conditions for the **E** and **D** fields, in order to show that the amplitudes of the reflected (E_R) and the transmitted (E_T) fields are given by

$$E_R = -\frac{(n_1/n_2)\cos\beta - \cos\alpha}{(n_1/n_2)\cos\beta + \cos\alpha} E_I, \qquad E_T = \frac{2\cos\alpha}{(n_2/n_1)\cos\alpha + \cos\beta} E_I.$$

The refractive indices are n_1 and n_2, and β is the angle between the transmitted wave and the y axis.

b) Calculate the Brewster angle $\alpha = \alpha_b$, for which we have $E_R/E_I = 0$.

c) For the case $n_1 > n_2$, determine the limiting angle $\alpha = \alpha_g$ at which total refraction occurs ($\beta = \pi/2$). Show furthermore for $\alpha > \alpha_g$ that we have $E_R/E_I = \exp(-2i\phi)$, meaning that total refraction leads to a phase shift of -2ϕ. Find ϕ.

Problem 3: Dielectric-Metal Contact

Consider the same plane wave as in problem 2, but for normal incidence $\alpha = 0$. The second medium ($y > 0$) is now a metal, i.e. $n_2 = n + i\gamma$ and $k_2 = k + i\kappa$.

a) Using the relation between **H** and **E** field to show that the coefficients for refraction and transmission are given by

$$\frac{H_T}{H_I} = \frac{2(n+i\gamma)}{(n+i\gamma)+n_1}, \qquad \frac{E_T}{E_I} = \frac{2n_1}{(n+i\gamma)+n_1}$$
$$\frac{H_R}{H_I} = \frac{(n+i\gamma)-n_1}{(n+i\gamma)+n_1}, \qquad \frac{E_R}{E_I} = \frac{(n+i\gamma)-n_1}{(n+i\gamma)+n_1}$$

b) In a good conductor we have $\sigma >> \epsilon\omega$. Use equations $k^2 = \frac{\mu\epsilon\omega^2}{2}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right)$ $\kappa^2 = \frac{\mu\epsilon\omega^2}{2}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)$ to show $k_2 \approx (1+i)\sqrt{\frac{\mu\omega\sigma}{2}}$ in this limit. Show furthermore that $E_T/E_I = 2k_1/(k_2+k_1) \to 0$, but H_T/H_I stays finite. Discuss H_T/E_T (phase!).

c) Calculate the transmittivity T (ratio of transmitted to incoming intensity), and the reflectivity $R = 1 - T$. Show in the limit $\sigma \to \infty$ that we have $T \to 0$ and $R \to 1$.