Elektrodynamics Exercises WS 2015-16

Problem set 3, 20.11.2015

Problem 1: Wave guide

Consider a wave conductor with rectangular shape with dimensions a and b, which is infinitely long in z-direction. The lateral faces are perfectly conducting.

a) Show, using boundary conditions that

$$\mathbf{E} = \begin{pmatrix} \alpha \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \\ \beta \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \\ \gamma \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \end{pmatrix} e^{i(kz-\omega t)} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} \alpha' \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \\ \beta' \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \\ \gamma' \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \end{pmatrix} e^{i(kz-\omega t)}$$

where $n, m \in N_0$.
Hint: Use the following Ansatz for each component of the electric and magnetic fields: $R_\alpha(\mathbf{r}, t) = g_\alpha(x) h_\alpha(y) e^{ikz - i\omega t}$, with $g_\alpha(x) = a_\alpha \cos(k_x x) + b_\alpha \sin(k_x x)$ and $h_\alpha(y) = c_\alpha \cos(k_y y) + d_\alpha \sin(k_y y)$.

b) (**2 points**) Show based on Maxwell's equations that constants α, β, γ, α', β' and γ' depend only on two free parameters δ and δ' and check that

$$\alpha = \frac{n\pi}{a} k\delta + \frac{m\pi}{b} \frac{\omega}{c} \delta' \qquad \alpha' = \frac{n\pi}{a} k\delta' - \frac{m\pi}{b} \frac{\omega}{c} \delta$$

$$\beta = \frac{m\pi}{b} k\delta - \frac{n\pi}{a} \frac{\omega}{c} \delta' \qquad \beta' = \frac{m\pi}{b} k\delta' + \frac{n\pi}{a} \frac{\omega}{c} \delta$$

$$\gamma = -i \left(\frac{\omega^2}{c^2} - k^2 \right) \delta \qquad \gamma' = i \left(\frac{\omega^2}{c^2} - k^2 \right) \delta'$$

Hint: To solve the problem you need the solution of the wave equation.

Problem 2: Dielectric function $\varepsilon(\omega)$

We want to calculate the dielectric function in a simplified model. Let us consider x-ray radiation interacting with matter. At the length scale of the x rays, we assume that the electrons behave as free particle.

a) Use the equation of motion of free electrons in an electric field and the Ansatz for a plane wave $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, and calculate the relation between current density $\mathbf{j}$ and field $\mathbf{E}$.

b) By comparison of the microscopic Maxwell equation

$$\nabla \times \mathbf{B} - \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

with the corresponding macroscopic Maxwell equation

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}_f$$

we can determine now the dielectric function. For that purpose you have to use $\mathbf{D} = \varepsilon(\omega)\mathbf{E}$ and $\mu = \mu_0$. Show that we get an equation in analogy to Eq. (14.11) of the lecture notes.

c) Suppose now that the x-ray radiation hits the surface of matter with the above determined dielectric function. Determine the angle of total reflection as function of frequency ω.