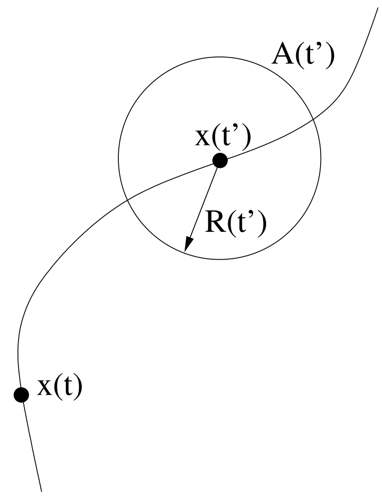
Elektrodynamics Exercises WS 2015-16

Problem set 4, 04.12.2015

Problem 1: Radiation of a charged particle



Eq. (15.72) in the lecture notes defines the Poynting vector for the radiation field:

$$\mathbf{S}_s = \left[\frac{q^2 \mathbf{N}}{16\pi^2 \epsilon_0 c^3 \kappa^6 R^2} (\mathbf{N} \times [(\mathbf{N} - \mathbf{v}/c) \times \mathbf{a}])^2 \right]_{ret}$$

where $\mathbf{N} = \mathbf{R}/R$. The energy that flows per time unit through the surface element $d\mathbf{A}(t')$ is given by $\mathbf{S}_s(t') \cdot d\mathbf{A}(t')dt$. Because all properties are expressed in the retarded time t', and not in t, we have to transform all quantities from dt to dt'. This gives the flow of energy as $dP(t')dt' = \mathbf{S}_s(t') \cdot d\mathbf{A}(t')(dt/dt')dt'$.

a) Calculate the factor (dt/dt') and show that

$$\frac{dP}{d\Omega} = \left[\frac{\mu_0 q^2}{(4\pi)^2 c} \cdot \frac{|\mathbf{N} \times [(\mathbf{N} - \mathbf{v}/c) \times \mathbf{a}]|^2}{\kappa^5} \right]_{ret}$$

with $d\Omega(t')$ the element of the solid angle, $\kappa = 1 - \mathbf{N} \cdot \mathbf{v}/c$. The retarded time t' is determined by the condition $|\mathbf{R}(t')| = c(t - t')$.

Consider now the following cases:

b) Slow Motion. Let the θ be the angle between the acceleration $\mathbf{a}$ and $\mathbf{R}$. Consider now small velocities $v \ll c$, $\kappa \approx 1$, and

b1) show that $\frac{dP}{d\Omega} = \left[\frac{\mu_0 q^2}{(4\pi)^2 c} |\mathbf{a}|^2 \sin^2\theta \right]_{ret}$.

b2) integrate $\frac{dP}{d\Omega}$ over the solid angle. Show that $P = \left[\frac{\mu_0 q^2}{6\pi c} |\mathbf{a}|^2 \right]_{ret}$.

c) Linear Motion Show that the power depends in the following way on the angle:

$$\frac{dP}{d\Omega} = \left[\frac{\mu_0 q^2}{(4\pi)^2 c} |\mathbf{a}|^2 \frac{\sin^2\theta}{(1 - v \cdot \cos\theta / c)^5} \right]_{ret}$$

where θ is again angle between $\mathbf{a}$ and $\mathbf{R}$. Take into account that $\mathbf{v}$ and $\mathbf{a}$ are parallel.

c1) Show that for $v \ll c$ you get back equation (b1).

c2) Show that the maximum of $dP/d\Omega$ is reached for $\cos\theta_{max} = \frac{\sqrt{1+15(v/c)^2}-1}{3(v/c)}$.
Hint: Substitute $\cos\theta \equiv x$ and find maximum with respect to x.

c3) Do an expansion of above equation up to second order in $\beta = v/c \ll 1$ and show $\theta_{max} = \frac{\pi}{2} - \frac{5}{2}\beta$.
Hint: Please use Taylor expansion of the cosine at $\theta = \pi/2$.

c4) Expand now $\cos\theta_{max}$ in terms of γ^{-1} up to γ^{-2} for the case $v \approx c$, $\gamma^{-2} = 1-(v/c)^2 \ll 1$. Show that $\theta_{max} = \frac{1}{2}\gamma^{-1}$. What does this result mean for high (relativistic) velocities, in what direction does the particle radiate?
Hint: Please use Taylor expansion of the cosine at $\theta = 0$.

c5) Calculate the power P by integrating over the solid angle $d\Omega$ and show that $P = \left[\frac{\mu_0 q^2}{6\pi c}|\mathbf{a}|^2\gamma^6\right]_{ret}$.

d) Circular motion. Show that the angular dependence of the radiated power is

$$\frac{dP}{d\Omega} = \left[\frac{\mu_0 q^2}{(4\pi)^2 c}\frac{|\mathbf{a}|^2}{(1-v\cdot\cos\theta/c)^3}[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-v\cdot\cos\theta/c)^2}]\right]_{ret}.$$

Here θ and ϕ are polar and azimuthal angles respect to $\mathbf{v}$. Find the values of θ, ϕ, for which $dP/d\Omega$ reaches its maximum. Show that this radiation is in forward direction, i.e. in the direction of the velocity $\mathbf{v}$ of the particle (synchrotron radiation). Take into account that $\mathbf{v}$ and $\mathbf{a}$ are orthogonal vectors.

Problem 2: Dipole antenna

We consider an antenna with the following current density (I_0 is a constant):

$$\mathbf{j} = I_0 \text{ Re } \delta(x)\delta(y)\theta(L/2-|z|)\mathbf{e}_z e^{-i\omega t}.$$

a) Calculate the vector potential $\mathbf{A}(\mathbf{r},t)$ (in Lorentz gauge) to leading order in $1/r$. Note that L/λ ($\lambda = 2\pi c/\omega$) is not small!
Hint: You can substitute $|\mathbf{r}-\mathbf{r}'|$ in the denominator by $|\mathbf{r}|$. In the exponent, however, you have to use $|\mathbf{r}-\mathbf{r}'| \approx |\mathbf{r}| - \mathbf{r}'\cdot\mathbf{r}/r$.

b) Calculate the field $\mathbf{B}(\mathbf{r},t)$, the Poynting vector and its time average $\bar{\mathbf{S}}$.
Hint: for large $\mathbf{r}$, $\nabla \to ik\mathbf{N}$, with $\mathbf{N} = (\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$.

c) Determine the angular dependence of $\bar{\mathbf{S}}$ for a $\lambda/2$ antenna, i.e. with $L = \lambda/2$. Show furthermore that for short antennas, i.e. $L \ll \lambda$, the angular dependence is given by the characteristic of a Hertzian dipole (Eq. 16.24 in the lecture notes):

$$\bar{\mathbf{S}} = \frac{\mu_0}{16\pi^2 c}\omega^4 d^2 \frac{\sin^2\theta}{2r^2}\mathbf{N}.$$

d) Calculate for a long antenna, $L \gg \lambda$, the total power of radiation to infinity.